Long-time self-similar asymptotic of the macroscopic quantum models

Hai-Liang Li[1], Guo-Jing Zhang[2], Min Zhang[1], Chengchun Hao[3]

[1] *Department of Mathematics, Capital Normal University, Beijing 100037, P.R.China*

[2] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P.R.China*

[3] *Institute of Mathematics, Academy of Mathematics & Systems Science, CAS, Beijing 100190, P.R.China*

E-mail: hailiang.li.math@gmail.com (H.L), zhanggj112@nenu.edu.cn (G.Z)

zhangminzi2004@163.com (M.Z), hcc@amss.ac.cn (C.H)

Abstract

The unipolar and bipolar macroscopic quantum models derived recently for instance in the area of charge transport are considered in spatial one-dimensional whole space in the present paper. These models consist of nonlinear fourth-order parabolic equation for unipolar case or coupled nonlinear fourth-order parabolic system for bipolar case. We show for the first time the self-similarity property of the macroscopic quantum models in large time. Namely, we show that there exists a unique global strong solution with strictly positive density to the initial value problem of the macroscopic quantum models which tends to a self-similar wave (which is not the exact solution of the models) in large time at an algebraic time-decay rate.

1 Introduction

The quantum hydrodynamic (QHD) model for semiconductors is derived and studied recently in the modelings and simulations of semiconductor devices, where the effects of quantum mechanics arise. The basic observation concerning the quantum hydrodynamics is that the energy density consists of one additional new quantum correction term of the order $O(\varepsilon)$ introduced first by Wigner [27] in 1932, and that the stress tensor contains also an additional quantum correction part [1] related to the quantum Bohm potential [3]

$$Q(\rho) = -\frac{\varepsilon^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}, \tag{1.1}$$

with observable $\rho > 0$ the density, m the mass, and ε the Planck constant. The quantum potential Q is responsible for producing the quantum behavior. Such possible relation was also implied in the original idea initialized by Madelung [25] to derive quantum fluid-type equations

in terms of Madelung's transformation applied to wave functions of the Schrödinger equation of the pure state. Recently, the moment method is employed to derive quantum hydrodynamic equations for semiconductor device at nano-size based on the Wigner-Boltzmann (or quantum Liouville) equation, refer to [13] for details. For more important progress on the derivation of macroscopic quantum models in terms of the entropy minimizer principle, one can refer to the recent interesting works [7, 13, 15, 16] and the references therein.

Starting with the quantum hydrodynamical models and performing the relaxation limit asymptotical analysis, the macroscopic quantum (Drift-Diffusion) model is derived rigorously [24] for the model of the unipolar carrier, the methods employed therein can be generalized to general bipolar carriers. For positive charge density, these models are indeed nonlinear fourth-order parabolic equation for the unipolar case or the coupled nonlinear fourth-order parabolic system for the bipolar case. We would like to mention that according to the recent result in [6], the model (1.4) can also be viewed as a relaxation limiting equation of the quantum fluid model which can be derived by the nonlinear Schrödinger-Langevin equation, for which the rigorous short time existence of weak solutions is proven recently in [20].

We are interested in the long time asymptotical behavior of solutions to the macroscopic quantum models in the present paper in the one-dimensional real line, and we shall show that the global classical solutions to the IVP (1.2)–(1.3) and the IVP (1.4)–(1.5) admit the character of self-similarity in large time. In general, the typical bipolar macroscopic quantum model widely used in semiconductor modeling in one dimension is the following coupled nonlinear parabolic system

$$\begin{cases} \partial_t \rho_i - p(\rho_i)_{xx} + \varepsilon^2 \left(\rho_i \left(\dfrac{(\sqrt{\rho_i})_{xx}}{\sqrt{\rho_i}} \right)_x \right)_x + (-1)^{i+1} (\rho_i E)_x = 0, \quad i = a, b, \\ E_x = \rho_a - \rho_b, \qquad t > 0, \ x \in \mathbb{R}, \end{cases} \tag{1.2}$$

together with the initial data

$$\rho_i(x,0) = \rho_{i,0}(x) > 0, \quad x \in \mathbb{R}, \quad \rho_{i,0}(\pm\infty) = \rho_\pm > 0, \quad i = a, b, \tag{1.3}$$

where ρ_a, $\rho_b > 0$ denote the macroscopic densities for electron and hole respectively [13], $p(\rho_i)$ is the pressure function depending on the density ρ_i, $\varepsilon > 0$ is the scaled Planck constant, and E denotes the self-consistent electric field. We also use the symbols $(-1)^a = -1$ and $(-1)^b = 1$ for the simplicity of statements.

In the absence of the electric filed (or in the so-called quasi-neutral domain), the initial value (IVP) problem (1.2) reduces to the IVP problem for the following unipolar macroscopic

quantum model

$$\partial_t\rho - p_{xx} + \varepsilon^2\Big(\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x\Big)_x = 0, \quad t>0 \tag{1.4}$$

$$\rho(x,0)=\rho_0(x)>0,\ x\in\mathbb{R}, \quad \rho_0(\pm\infty)=\rho_\pm>0, \tag{1.5}$$

where $\rho=\rho(x,t)>0$ is the density of electron or hole and $p=p(\rho)$ is the pressure function depending on ρ. It should be noted that a similar model (the DDLS model), which takes the form of (1.4) but without the pressure term (i.e., $p=0$), also arises in the study of interface fluctuations in spin systems, for instance [8].

There are recently many analysis results on macroscopic quantum models of the fourth-order parabolic type (1.2) or (1.4) and related models. For the Eq. (1.4) without the density pressure function term (the DLSS model [8]), the positive classical solutions are proven locally in-time in one-dimensional periodic domain [2], and the global existence of a spatially periodic H^1 solution and its exponential convergence to an equilibrium state is shown [4] in terms of the entropy method and the Csiszar-Kullback inequality for "small" initial data. This is mainly due to the failure of the maximum principle which makes it impossible to establish a-priorily the upper and lower bounds of the density and obtain the global in-time existence of solutions with the strictly positive density. This, however, leads to the interesting results on the global existence of a nonnegative weak solution, which is first established in a one-dimensional bounded domain with the Dirichlet and Neumann boundary condition [14] where an interesting entropy estimate is introduced to show the global existence. Since then, some additional first order entropies are also obtained [16, 21]. More recently, the global existence of multi-dimensional nonnegative weak solutions and their exponential decay to an equilibrium state is also shown for the DLSS model in a periodical domain [17] based on the extended multi-dimensional algorithmic entropy construction argument, and for the DLSS model with an additional term of the given drift potential [11] in the framework of variation and Wasserstein's metric subject to the finite initial mass. For more analysis related to the DLSS model or Eq. (1.4) about numerical simulations or long time convergences, one can refer to the recent papers [5, 12, 18, 23] and references therein. As for the bipolar quantum model (1.2), the existence of a stationary state is only analyzed recently [26]. Some interesting quasi-neutral limit has been analyzed recently [22].

However, there are few results on the global existence of classical (strong) solutions with the strictly positive density and the long time asymptotical behaviors of classical solutions for the macroscopic quantum models (1.2) and (1.4) in the whole spatial space, although there are a short time classical solution with the positive density for the DLSS model [2] and a global existence for nonnegative weak solutions [14, 16, 21, 11, 18]. The main difficulties in dealing with the macroscopic quantum models (1.2) and (1.4) consist of the strong nonlinearity,

the degeneracy at vacuums, and the failure of the maximum principle, and the coupling and interaction between the two carriers for the bipolar case.

It should be noted that it seems not obvious how to generalize the framework of entropy estimates and/or Wasserstein's metric, used for instance in [11, 12, 16] to establish the global existence of nonnegative solutions with the finite initial mass, to show the global existence of a classical solution with the strictly positive density for Eqs. (1.2) or (1.4) in the whole spatial space subject to the infinite initial mass (the case to be dealt with in the present case), since in general Poincaré's inequality failed and it is not trivial to establish the a-priori uniform control of the density with respect to the time in order to understand the long time behavior of global solutions.

In this paper, we are interested in the large time asymptotical behavior of the solution to the IVP (1.2)–(1.3) for the bipolar case and the IVP (1.4)–(1.5) for the unipolar case, and we shall show that the global classical solutions to IVP (1.2)–(1.3) and IVP (1.4)–(1.5) admit the character of self-similarity in large time so long as it is around the self-similar wave initially. To be more precise, let's introduce the quasi-linear parabolic equation

$$\rho_t = p(\rho)_{xx}, \quad p'(\rho) > 0. \tag{1.6}$$

It is well-known that Eq.(1.6) has a unique self-similar solution $W(x,t)$ up to a position shift (see[9])

$$\rho(x,t) =: W(\xi), \quad \xi = \frac{x}{\sqrt{t+1}}, \quad W(\pm\infty) = \rho_\pm, \tag{1.7}$$

and the solution $W(\xi)$ is increasing if $\rho_- < \rho_+$ and decreasing if $\rho_- > \rho_+$, and $W_\xi \to 0$ as $|\xi| \to \infty$.

To begin with, let us consider the case of regular initial data and take the initial datum of the IVP (1.2)–(1.3) and IVP (1.4)–(1.5) close to the self-similar solution W in some Sobolev norm. Then, we show in terms of the energy method below that there exists a global unique classical solution to the IVP (1.2)–(1.3) or the IVP (1.4)–(1.5), which in particular tends to the self-similar wave W in large time with an algebraic decay rate.

We should also mention that it is not obvious so far how to generalize the entropy functional or Wasserstein's metric technique as used in [11, 12, 16, 23] to prove the convergence of global nonnegative solutions of the Eqs. (1.2) and (1.4) to the self-similar wave since the self-similar wave W is not a solution of the fourth-order quantum models (1.2) and (1.4) and the convergence itself is a singular process in large time. Moreover, some additional information on the lower and upper bounds of the density are needed to prevent the possible appearance of the singularity of the fourth order differential operator near vacuums. This is nontrivial however due to the failure of the maximum principal theory for the fourth-order equation. It is also interesting to whether

or not the global classical solution of the quantum models (1.2) and (1.4) shall converge to the self-similar wave W in large time for general initial data, instead of the small perturbation of the self-similar wave, it is left for further investigation.

We first investigate the long time asymptotical behavior of global solutions to the IVP for the unipolar equation (1.4)–(1.5), and then discuss the corresponding IVP for the bipolar model (1.2)–(1.3).

Let

$$z_0(x) = \int_{-\infty}^{x} (\rho_0(y) - W(y + x_0))dy, \tag{1.8}$$

then

$$z_{0x} = \rho_0(x) - W(x + x_0),$$

where x_0 is determined by

$$\int_{-\infty}^{+\infty} (\rho_0(x) - W(x + x_0))dx = 0.$$

For the unipolar case, our main result on the global solution and its large time behavior of IVP (1.4)–(1.5) is given as follows.

Theorem 1.1 *Let $p'(\rho) > 0$ for $\rho > 0$. Assume that $\delta =: |\rho_+ - \rho_-| \ll 1$ and $z_0 \in H^3(\mathbb{R})$ with $\delta_0 =: \|z_0(x)\|_{H^3(\mathbb{R})}$ small enough, then there is a unique global strong solution $\rho > 0$ of IVP* (1.4)-(1.5) *such that*

$$\rho - W \in L^\infty([0, \infty); H^2(\mathbb{R})) \cap L^2([0, \infty); H^4(\mathbb{R})),$$

and the solution ρ converges to the self-similar wave $W(\frac{x+x_0}{\sqrt{t+1}})$ of the Eq. (1.6) *with an algebraic time decay rate*

$$\|\partial_x^k(\rho - W)(t)\|_{L^2(\mathbb{R})} \leqslant C(1 + t)^{-\frac{k+1}{2}}, \qquad k = 0, 1, 2,$$
$$\|(\rho - W)(t)\|_{L^\infty(\mathbb{R})} \leqslant C(1 + t)^{-\frac{3}{4}},$$

where C is a positive constant dependent of δ and δ_0.

Next, we state the main result on the convergence to the self-similar wave for the bipolar case. Although it leads to additional difficulties, the coupling between carriers in (1.2) may cause some cancelation, and it is not clear that both the densities of the bipolar QDD (1.2) behave still or not like the unipolar one for a small perturbation.

Denotes

$$z_0^i(x) = \int_{-\infty}^{x} (\rho_{i,0}(y) - W(y + x_0))dy, \quad i = a, b, \tag{1.9}$$

then

$$z^i_{0x} = \rho_{i,0}(x) - W(x+x_0), \quad i = a, b,$$

where x_0 is determined by

$$\int_{-\infty}^{+\infty} (\rho_{i,0}(x) - W(x+x_0))dx = 0.$$

Theorem 1.2 *Let $p'(\rho) > 0$ for $\rho > 0$, and $\delta =: |\rho_+ - \rho_-| \ll 1$. Assume that $\inf_{x\in\mathbb{R}} \rho_{i,0} > 0$ $(i = a, b)$ with $\int(\rho_{a,0} - \rho_{b,0})dx = 0$, and $z^i_0 \in H^3(\mathbb{R})$ with $\delta_0 =: \|z^a_0\|_{H^3(\mathbb{R})} + \|z^b_0\|_{H^3(\mathbb{R})}$ small enough, then there is a unique global strong solution (ρ_a, ρ_b, E) of the IVP (1.2)-(1.3) with $\rho_a > 0$, $\rho_b > 0$ such that*

$$\rho_i - W \in L^\infty([0,\infty); H^2(\mathbb{R})) \cap L^2([0,\infty); H^4(\mathbb{R})), \quad i = a, b,$$

and both ρ_a and ρ_b converge to the self-similar wave $W(\frac{x+x_0}{\sqrt{t+1}})$ of the Eq. (1.6) with an algebraic time decay rate

$$\begin{aligned} \|\partial^k_x(\rho_a - W)(t)\|_{L^2(\mathbb{R})} + \|\partial^k_x(\rho_b - W)(t)\|_{L^2(\mathbb{R})} &\leqslant C(1+t)^{-\frac{k+1}{2}}, \qquad k = 0,1,2, \\ \|(\rho_a - W)(t)\|_{L^\infty(\mathbb{R})} + \|(\rho_b - W)(t)\|_{L^\infty(\mathbb{R})} &\leqslant C(1+t)^{-\frac{3}{4}}, \\ \|E(t)\|_{H^1(\mathbb{R})} &\leqslant Ce^{-\beta t}, \end{aligned}$$

where $C > 0$ and $\beta > 0$ are constants dependent of δ and δ_0.

Notations. $L^p(\mathbb{R})$ and $H^k(\mathbb{R})$ denote the usual Lebesgue integrable functions space and the Sobolev space with norm $\|\cdot\|_{L^p(\mathbb{R})}$ and $\|\cdot\|_{H^k(\mathbb{R})}$ respectively. we also use $\|\cdot\|$ to denote $\|\cdot\|_{L^2(\mathbb{R})}$ for simplicity. C and c are used to denote general positive constants.

2 Proof of main results

We shall prove Theorems 1.1-1.2 in this section. The key is to establish the a-priori estimates for short time strong solutions. Without the loss of generality, we establish the expected estimates in order to prove Theorem 1.1 in Sect. 2.1, and show how to derive the estimates about the electric filed in the proof of Theorem 1.2 in Sect. 2.2.

2.1 The unipolar case

In this section, we shall transform the primary equations in order to study the existence and in particular its large time behavior of the global solutions of the IVP (1.4)-(1.5).

Denote

$$z(x,t) = \int_{-\infty}^{x} \left(\rho(y,t) - W\left(\frac{y+x_0}{\sqrt{t+1}}\right)\right) dy, \tag{2.1}$$

then

$$z_x = \rho(x,t) - W(\frac{x+x_0}{\sqrt{t+1}}). \tag{2.2}$$

We will derive the fourth order parabolic equation for z. Since W satisfies

$$W_t = p(W)_{xx}, \tag{2.3}$$

and $\rho = \rho(x,t)$ satisfies Eq.(1.4), we have

$$(\rho - W)_t - (p(\rho) - p(W))_{xx} + \varepsilon^2 \left(\rho \left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x\right)_x = 0. \tag{2.4}$$

Integrating (2.4) over $(-\infty, x)$ with respect to the spatial variable and assuming $\rho_{xx} \to 0$ as $|x| \to \infty$, we obtain from (2.1) and (2.2) the parabolic equation of the fourth order for z of the following form

$$z_t - (p'(W)z_x)_x + \frac{\varepsilon^2}{2} z_{xxxx} = (f_1 + f_2)_x, \tag{2.5}$$

with the initial datum

$$z(x,0) = z_0(x), \tag{2.6}$$

where

$$f_1 = \frac{\varepsilon^2}{2}\frac{(W_x + z_{xx})^2}{W + z_x} - \frac{\varepsilon^2}{2} W_{xx}, \tag{2.7}$$

$$f_2 = p(z_x + W) - p'(W)z_x - p(W). \tag{2.8}$$

Note that we have used the fact

$$\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x = \frac{1}{2}\rho_{xxx} - \frac{1}{2}\left(\frac{\rho_x^2}{\rho}\right)_x.$$

The existence of the global solution and the large time behavior for the IVP (2.5)–(2.6) is obtained by the following proposition.

Proposition 2.1 *Let $p'(\rho) > 0$ for $\rho > 0$ and $\delta = |\rho_+ - \rho_-| \ll 1$. Assume that $\|z_0\|_{H^3(\mathbb{R})} \leqslant \delta_0$ with $\delta_0 > 0$ sufficiently small, then there is a unique global strong solution z to the IVP* (2.5)-(2.6) *satisfying*

$$z \in L^\infty([0,\infty); H^3(\mathbb{R})) \cap L^2([0,\infty); H^5(\mathbb{R})),$$
$$\|\partial_x^k z(\cdot,t)\|_{L^2(\mathbb{R})} \leqslant C(\delta,\delta_0)(1+t)^{-\frac{k}{2}}, \quad k = 0,1,2,3,$$
$$\|\partial_x^k z(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqslant C(\delta,\delta_0)(1+t)^{-\frac{1+2k}{4}}, \quad k = 0,1,2,$$

where $C(\delta,\delta_0) > 0$ is a positive constant depending only on δ and δ_0.

Remark 2.2 *It is sufficient to prove Theorem 1.1 in terms of Proposition 2.1 due to the relation between ρ and z*

$$\rho = W + z_x$$

and the transformation of (2.2)–(2.5). The positivity of ρ can be assured by the positivity of W and the smallness of z_x. From the above, we also have

$$\|\partial_x^k(\rho - W)(t)\|_{L^2(\mathbb{R})} = \|\partial_x^{k+1} z(t)\|_{L^2(\mathbb{R})}, \quad k = 0, 1, 2.$$

In order to prove Proposition 2.1, let us assume that for the local in-time solution and $T > 0$

$$\delta_T = \max_{k=0,1,2,3} \sup_{0 \leqslant t \leqslant T} (1+t)^{\frac{k}{2}} \|\partial_x^k z\| \ll 1. \tag{2.9}$$

By the Nirenberg's inequality and the above assumption, we have

$$\|\partial_x^k z\|_{L^\infty(\mathbb{R})} \leqslant c\delta_T (1+t)^{-\frac{2k+1}{4}}, \quad k = 0, 1, 2. \tag{2.10}$$

The theorem for the existence of the local in-time solution is

Theorem 2.3 *Let $p'(\rho) > 0$ for $\rho > 0$ and assume that $\|z_0\|_{H^3(\mathbb{R})}$ small enough and $\inf_{x\in\mathbb{R}}(z_{0x} + W(x + x_0)) > 0$ (i.e. $\inf_{x\in\mathbb{R}} \rho_0 > 0$). Then there exists a $T^* > 0$ such that there is a unique local solution of the IVP (2.5)-(2.6) for $t \in (0, T^*)$ satisfying*

$$\|z(\cdot, t)\|_{H^3(\mathbb{R})} < \infty, \quad \rho(x, t) = z_x(x, t) + W(x + x_0, t) > 0.$$

The proof of Theorem 2.3 can be obtained by a standard method (see [11]), we omit it.

Now, we list the L^p-estimates of the derivatives of W and $n = \sqrt{W}$ as follows.

Lemma 2.4 *Let W be the self-similar solution of (1.6) and $n = \sqrt{W}$, then it holds that (see[6])*

$$\|\partial_x^j W(\cdot, t)\|_{L^p(\mathbb{R})} \leqslant C\delta(1+t)^{-\frac{j}{2}+\frac{1}{2p}}, \tag{2.11}$$

$$\|\partial_x^j n(\cdot, t)\|_{L^p(\mathbb{R})} \leqslant C\delta(1+t)^{-\frac{j}{2}+\frac{1}{2p}}, \tag{2.12}$$

for $j \geq 0, p \in [1, +\infty]$, $C > 0$ is some constant.

For f_1, f_2, we have the following estimates.

Lemma 2.5 *Under the assumption (2.9), it holds for f_1, f_2*

$$f_1 = O(\delta_T + \delta) z_{xx} + O(\delta) r_2, \tag{2.13}$$

$$f_2 = O(\delta_T) z_x, \tag{2.14}$$

where the function $r_k(x,t)$ is related to the kth order derivative of W with respect to x, which satisfies by the definition

$$\|r_k(\cdot,t)\|_{L^p(\mathbb{R})} \leqslant C(1+t)^{-\frac{k}{2}+\frac{1}{2p}},\ k=0,1,2,\cdots,\ \text{and } p\in[1,+\infty]. \tag{2.15}$$

Proof. From Lemma 2.4, (2.7) and (2.8), we have

$$\begin{aligned} f_1 &= \frac{\varepsilon^2}{2}\frac{(W_x+z_{xx})^2}{W+z_x}-\frac{\varepsilon^2}{2}W_{xx} \\ &= O(1)(W_x^2+2W_x\cdot z_{xx}+z_{xx}^2+W_{xx}) \\ &= (2W_x+z_{xx})\cdot z_{xx}+O(\delta)r_2 \\ &= O(\delta_T+\delta)\cdot z_{xx}+O(\delta)r_2, \\ f_2 &= p(z_x+W)-p'(W)z_x-p(W) \\ &= \frac{p''(W)}{2}\cdot z_x^2+O(z_x^3) \\ &= Cz_x(z_x+z_x^2)=O(\delta_T)z_x, \end{aligned}$$

by which the proof is easy. $\square$

Lemma 2.6 *Under the assumption (2.9), it holds for the local in-time solution z*

$$\|z(t)\|^2+\int_0^t\|z_x(s)\|^2ds+\int_0^t\|z_{xx}(s)\|^2ds\leqslant O(\delta+\delta_0)^2, \tag{2.16}$$

for $0\leqslant t\leqslant T$, provided that $\delta_T+\delta$ is small enough.

Proof. Taking the L^2-inner product of (2.5) with z, we get with the help of the integration by parts

$$\begin{aligned} \int_{\mathbb{R}}[z_t-(p'(W)z_x)_x+\frac{\varepsilon^2}{2}z_{xxxx}]\cdot zdx &= \int_{\mathbb{R}}z_t\cdot z+p'(W)z_x^2+\frac{\varepsilon^2}{2}z_{xx}^2dx \\ &= \frac{1}{2}\frac{d}{dt}\|z\|^2+\frac{\varepsilon^2}{2}\|z_{xx}\|^2+\int_{\mathbb{R}}p'(W)z_x^2dx. \end{aligned}$$

By Lemma 2.5 for f_1, f_2 and Cauchy's inequality, we get

$$\begin{aligned} \int_{\mathbb{R}}(f_1+f_2)_x\cdot zdx &= \int_{\mathbb{R}}-(f_1+f_2)\cdot z_xdx \\ &\leqslant \alpha\|z_x\|^2+O(1)(\|f_1\|^2+\|f_2\|^2) \\ &\leqslant \alpha\|z_x\|^2+O(\delta_T+\delta)^2(\|z_{xx}\|^2+\|z_x\|^2)+O(\delta^2)(1+t)^{-\frac{3}{2}}, \end{aligned}$$

where $\alpha>0$ is a constant such that $\alpha+O(\delta_T+\delta)^2\leqslant\frac{1}{10}\inf\limits_{x\in\mathbb{R}}p'(W)$. Combining these estimates, we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+(\frac{\varepsilon^2}{2}-O(\delta_T+\delta)^2)\|z_{xx}\|^2+\int_{\mathbb{R}}(p'(W)-\alpha-O(\delta_T+\delta)^2)z_x^2dx\leqslant O(\delta^2)(1+t)^{-\frac{3}{2}},$$

Since $\delta+\delta_T \ll 1$, we have $\frac{\varepsilon^2}{2}-O(\delta+\delta_T)=O(\varepsilon^2)$ and

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+O(\varepsilon^2)\|z_{xx}\|^2+\int_{\mathbb{R}}(p'(W)-\alpha-O(\delta_T+\delta))z_x^2dx \leqslant O(\delta^2)(1+t)^{-\frac{3}{2}}.$$

Integrating the above inequality with respect to the time from 0 to t, we get

$$\|z(t)\|^2+\int_0^t\|z_{xx}(s)\|^2ds+\int_0^t\int_{\mathbb{R}}p'(W)z_x^2\,dxds \leqslant O(\delta_0+\delta)^2.$$

This, together with the fact $p'(W)>0$ for $W>0$, gives (2.16). □

Lemma 2.7 *Under the assumption (2.9), it holds for the local in-time solution z*

$$\|z_x(t)\|^2+\int_0^t\|z_{xx}(s)\|^2ds+\int_0^t\|z_{xxx}(s)\|^2ds \leqslant O(\delta+\delta_0)^2, \tag{2.17}$$

$$(1+t)\|z_x(t)\|^2+\int_0^t(1+s)(\|z_{xx}(s)\|^2+\|z_{xxx}(s)\|^2)ds \leqslant O(\delta+\delta_0)^2 \tag{2.18}$$

for $0 \leqslant t \leqslant T$ provided that $\delta_T+\delta$ is small enough.

Proof. Differentiating the equation (2.5) with respect to x and taking the L^2-inner product of the resulting equation with z_x, we get in view of the integration by parts

$$\begin{aligned}&\int_{\mathbb{R}}(f_1+f_2)_{xx}z_xdx\\ =&\int_{\mathbb{R}}[(z_t)_x-((p'(W)z_x)_x)_x+\frac{\varepsilon^2}{2}(z_{xxxx})_x]\cdot z_xdx\\ =&\frac{1}{2}\frac{d}{dt}\|z_x\|^2+\frac{\varepsilon^2}{2}\|z_{xxx}\|^2+\int_{\mathbb{R}}p'(W)z_{xx}^2dx+l_1,\end{aligned} \tag{2.19}$$

where

$$|l_1|=|\int_{\mathbb{R}}p''(W)W_xz_xz_{xx}dx| \leqslant O(1)\|W_x\|_{L^\infty}^2\|z_x\|^2+\alpha\|z_{xx}\|^2, \tag{2.20}$$

with the same α as in Lemma 2.6. By Lemma 2.6 and Cauchy's inequality, we get

$$\begin{aligned}|\int_{\mathbb{R}}(f_1+f_2)_{xx}z_xdx| =&|\int_{\mathbb{R}}(f_1+f_2)_xz_{xx}dx|\\ \leqslant&O(1)\|(f_1+f_2)_x\|^2+\alpha\|z_{xx}\|^2\\ \leqslant&O(\delta_T+\delta)(\|z_{xx}\|^2+\|z_{xxx}\|^2)+\alpha\|z_{xx}\|^2+O(\delta^2)(1+t)^{-\frac{5}{2}},\end{aligned} \tag{2.21}$$

where we have used the assumption (2.10) and a similar analysis as in Lemma 2.5 with the help of Lemma 2.4. Combining (2.19)–(2.21) and integrating the resulting inequality over $[0,t]$, we have after simplifying

$$\|z_x(t)\|^2+\int_0^t\|z_{xx}(s)\|^2ds+\int_0^t\|z_{xxx}(s)\|^2ds \leqslant O(\delta_0+\delta)^2,$$

which gives (2.17). To obtain (2.18), differentiating the equation (2.5) with respect to x and taking the L^2-inner product of the resulting equation with $(1+s)z_x$, similarly with the former we get after the integration by parts

$$\begin{aligned}\int_{\mathbb{R}}(f_1+f_2)_{xx}z_x(1+t)dx =& \int_{\mathbb{R}}[(z_t)_x-((p'(W)z_x)_x)_x+\frac{\varepsilon^2}{2}(z_{xxxx})_x]\cdot z_x(1+t)dx\\ =&\frac{1}{2}\frac{d}{dt}[(1+t)\|z_x\|^2]-\frac{1}{2}\|z_x\|^2\\ &+\frac{\varepsilon^2}{2}(1+t)\|z_{xxx}\|^2+\int_{\mathbb{R}}(1+t)p'(W)z_{xx}^2dx+(1+t)l_1, \end{aligned} \tag{2.22}$$

with the estimate

$$\begin{aligned}|(1+t)l_1| =&|\int_{\mathbb{R}}(1+t)p''(W)W_xz_xz_{xx}dx|\\ \leqslant& O(1)(1+t)\|W_x\|_{L^\infty}^2\|z_x\|^2+\alpha(1+t)\|z_{xx}\|^2\\ \leqslant& O(1)(\delta^2)\|z_x\|^2+\alpha(1+t)\|z_{xx}\|^2, \end{aligned} \tag{2.23}$$

where we have used Lemma 2.4 for $\|W_x\|_{L^\infty}^2\leqslant c\delta^2(1+t)^{-1}$. We also have estimates

$$\begin{aligned}\int_{\mathbb{R}}(1+t)(f_1+f_2)_{xx}z_xdx =&|\int_{\mathbb{R}}(1+t)(f_1+f_2)_xz_{xx}dx|\\ \leqslant& O(\delta_T+\delta)(1+t)(\|z_{xx}\|^2+\|z_{xxx}\|^2)+\alpha(1+t)\|z_{xx}\|^2\\ &+O(\delta^2)(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{2.24}$$

Combining (2.22)–(2.24) and integrating the resulting inequality over $[0,t_1]$ will give us after simplification

$$(1+t_1)\|z_x(t_1)\|^2+\int_0^{t_1}(1+t)(\|z_{xx}(t)\|^2+\|z_{xxx}(t)\|^2)dt\leqslant O(\delta+\delta_0)^2,$$

where we have applied the derived estimates in Lemma 2.6 to get that

$$\int_0^{t_1}\|z_x\|^2dt\leqslant O(\delta+\delta_0)^2$$

for $t_1\in[0,T]$. This gives (2.18). □

Based on Lemma 2.6 and Lemma 2.7, we can perform the estimates of the second and the third order derivatives of z in the same procedure and we have

Lemma 2.8 *Under the assumption* (2.9), *it holds for the local in-time solution* z

$$\begin{aligned}&(1+t)^2\|z_{xx}(t)\|^2+\int_0^t(1+s)^2(\|z_{xxx}(s)\|^2+\|z_{xxxx}(s)\|^2)ds\leqslant O(\delta+\delta_0)^2,\\ &(1+t)^3\|z_{xxx}(t)\|^2+\int_0^t(1+s)^3(\|z_{xxxx}(s)\|^2+\|z_{xxxxx}(s)\|^2)ds\leqslant O(\delta+\delta_0)^2\end{aligned}$$

for $0\leqslant t\leqslant T$ *provided that* $\delta_T+\delta$ *is small enough.*

Proof. We give the sketch of the proof. Performing $\int_0^{t_1}\int_{\mathbb{R}}(1+t)^2(2.5)_{xx}z_{xx}dxdt$, we can get as in Lemma 2.7

$$(1+t_1)^2\|z_{xx}(t)\|^2+\int_0^{t_1}(1+t)^2(\|z_{xxx}(s)\|^2+\|z_{xxxx}(s)\|^2)dt$$
$$\leqslant \int_0^{t_1}(1+t)^2\|z_x\|_{L^\infty}^2\|z_{xx}\|^2dt+\int_0^{t_1}(1+t)^2\|r_4\|^2dt, \tag{2.25}$$

with r_4 defined in Lemma 2.5 satisfying

$$\|r_4\|^2\leqslant c\delta^2(1+t)^{-\frac{7}{2}}.$$

By (2.10) we also have $\|z_x\|_{L^\infty}^2\leqslant c\delta_T^2(1+t)^{-1}$. Thus, by (2.25) and Lemma 2.7 we have

$$(1+t_1)^2\|z_{xx}(t)\|^2+\int_0^{t_1}(1+t)^2(\|z_{xxx}(s)\|^2+\|z_{xxxx}(s)\|^2)dt\leqslant O(\delta+\delta_0)^2. \tag{2.26}$$

Similarly, by performing $\int_0^{t_1}\int_{\mathbb{R}}(1+t)^3(2.5)_{xxx}z_{xxx}dxdt$ and with the help of all the derived a-priori estimates we can obtain

$$(1+t)^3\|z_{xxx}(t)\|^2+\int_0^{t}(1+s)^3(\|z_{xxxx}(s)\|^2+\|z_{xxxxx}(s)\|^2)ds\leqslant O(\delta+\delta_0)^2.$$

□

The proof of Proposition 2.1. The Lemmas 2.6–2.8 show that the local solution satisfies the uniform bounds for short time ($\delta\ll 1,\delta_0\ll 1$) when the initial perturbation is small enough. By using continuous methods, we can extend the local solution to be a global one, which also satisfies Lemmas 2.6–2.8 for any time. Then the existence of the global solution to the IVP (2.5)-(2.6) is proven. The Theorem 1.1 is a direct corollary of Proposition 2.1. □

2.2 The bipolar case

In this subsection, we prove Theorem 1.2 about the IVP (1.2)-(1.3) for the bipolar case. We also shall transform the primary equations in order to study the existence and in particular its large time behavior of global solutions of the IVP (1.2)-(1.3).

Denote

$$z^i(x,t)=\int_{-\infty}^{x}\left(\rho_i(y,t)-W\left(\frac{y+x_0}{\sqrt{t+1}}\right)\right)dy,\quad i=a,b, \tag{2.27}$$

which implies

$$z^i_x=\rho_i(x,t)-W\left(\frac{x+x_0}{\sqrt{t+1}}\right),\quad i=a,b. \tag{2.28}$$

Similar to the unipolar case, assuming $(\rho_i)_{xx} \to 0$ as $|x| \to \infty$, we can rewrite the equations (1.2) as

$$\begin{cases} z_t^i - (p'(W)z_x^i)_x + \dfrac{\varepsilon^2}{2} z_{xxxx}^i + (-1)^{i+1}(z_x^i + W)E = (f_{i,1} + f_{i,2})_x, \quad i = a, b, \\ E_t - (p'(W)E_x)_x + \dfrac{\varepsilon^2}{2} E_{xxxx} + 2WE = (f_{a,1} - f_{b,1} + f_{a,2} - f_{b,2})_x - (z_x^a + z_x^b)E, \end{cases} \tag{2.29}$$

with initial data

$$z^i(x,0) = z_0^i(x), \quad z_0^i(\pm\infty) = 0, \quad i = a, b, \tag{2.30}$$

where

$$\begin{aligned} E &= \int_{-\infty}^{x} (\rho_a(y,t) - \rho_b(y,t))dy = z^a - z^b, \\ f_{i,1} &= \frac{\varepsilon^2}{2} \frac{(W_x + z_{xx}^i)^2}{W + z_x^i} - \frac{\varepsilon^2}{2} W_{xx}, \quad i = a, b, \\ f_{i,2} &= p(z_x^i + W) - p'(W)z_x^i - p(W), \quad i = a, b, \end{aligned}$$

and we recall we make use of the symbols $(-1)^a = -1$ and $(-1)^b = 1$ for the simplicity of statements.

We have the following results:

Proposition 2.9 *Let $p'(\rho) > 0$ for $\rho > 0$ and $\delta = |\rho_+ - \rho_-| \ll 1$. Assume that $\|z_0^a(x)\|_{H^3(\mathbb{R})} + \|z_0^b(x)\|_{H^3(\mathbb{R})} \leqslant \delta_0$ with $\delta_0 > 0$ sufficiently small, then there is a unique global strong solution (z^a, z^b, E) to the IVP* (2.29)*-*(2.30) *satisfying*

$$\begin{gathered} (z^a, z^b) \in (L^\infty([0,\infty); H^3(\mathbb{R})) \cap L^2([0,\infty); H^5(\mathbb{R})))^2, \\ \|\partial_x^k (z^a, z^b)(\cdot, t)\|_{L^2(\mathbb{R})} \leqslant C(\delta, \delta_0)(1+t)^{-\frac{k}{2}}, \quad k = 0,1,2,3, \\ \|\partial_x^k (z^a, z^b)(\cdot, t)\|_{L^\infty(\mathbb{R})} \leqslant C(\delta, \delta_0)(1+t)^{-\frac{1+2k}{4}}, \quad k = 0,1,2, \\ \|E(t)\|_{L^2(\mathbb{R})} \leqslant C(\delta, \delta_0)e^{-\beta t}, \end{gathered}$$

where $C(\delta, \delta_0) > 0$ and β are some positive constants depending only on δ and δ_0.

The proof of Proposition 2.9 can be made in the similar fashion as Proposition 2.1. Here, we only show how to deal with the electric field and show its exponential decay. Let us assume that it holds for local in-time solutions that

$$\delta_T = \max_{\substack{k=0,1,2,3 \\ i=a,b}} \sup_{0 \leqslant t \leqslant T} (1+t)^{\frac{k}{2}} \|\partial_x^k z^i\| \ll 1, \quad \text{for } T > 0. \tag{2.31}$$

Similar to Lemma 2.5, we have the following properties for $f_{i,1}$ and $f_{i,2}$, whose proof is very similar to that of Lemma 2.5, we omit details.

Lemma 2.10 *Under the assumption* (2.31), *we have for* $i = a, b$ *and* $j = 0, 1, 2$, *that*

$$\partial_x^j f_{i,1} = O(\delta_T + \delta)\partial_x^{j+2} z^i + O(\delta) r_{j+2}, \quad \partial_x^j f_{i,2} = O(\delta_T)\partial_x^{j+1} z^i,$$
$$\partial_x^j (f_{a,1} - f_{b,1}) = O(\delta_T + \delta)(\partial_x^{j+1} E + \partial_x^{j+2} E), \quad \partial_x^j (f_{a,2} - f_{b,2}) = O(\delta_T)\partial_x^{j+1} E,$$

where the function $r_k(x,t)$ *is the same one as in Lemma 2.5.*

We also have the bipolar version of Lemmas 2.4–2.8 as similar ways as in the unipolar case. Indeed, we can obtain the following a-priori estimates.

Lemma 2.11 *Under the assumption* (2.31), *it holds for the strong solutions* (z^a, z^b, E) *that*

$$\left\|\partial_x^k (z^a, z^b)(t)\right\|^2 + \sum_{j=1}^{2} \int_0^t \left\|\partial_x^{k+j} (z^a, z^b)(s)\right\|^2 ds + \int_0^t \left\|\partial_x^k E(s)\right\|^2 ds \leqslant O(\delta + \delta_0)^2, \tag{2.32}$$

$$(1+t)^k \left\|\partial_x^k (z^a, z^b)(t)\right\|^2 + \sum_{j=1}^{2} \int_0^t (1+s)^k \left\|\partial_x^{k+j} (z^a, z^b)(s)\right\|^2 ds \leqslant O(\delta + \delta_0)^2, \tag{2.33}$$

for $k = 0, 1, 2, 3$, *and*

$$\left\|\partial_x^k E(t)\right\| \leqslant O(\delta + \delta_0) e^{-\beta_k t}, \ k = 0, 1, 2, \quad \textit{for some constants } \beta_0 > \beta_1, \beta_2 > 0, \tag{2.34}$$

for $t \in [0, T]$, *provided that* $\delta_T + \delta$ *is small enough.*

Proof. We only give a sketch of the proof here since it is very similar to the unipolar case. Taking the kth order derivative of the first equation in (2.29) with respect to the spatial variable x, and taking the L^2 inner product of the resulting equation with $\partial_x^k z^i$, we get for $k = 0, 1, 2, 3$, with the help of (2.31) and Lemma 2.4 and 2.10, that

$$\begin{aligned}
&\sum_{i=a,b} \frac{1}{2}\frac{d}{dt}\left\|\partial_x^k z^i\right\|^2 + \sum_{i=a,b} \int_{\mathbb{R}} p'(W)(\partial_x^{k+1} z^i)^2 dx + \frac{\varepsilon^2}{2}\sum_{i=a,b}\left\|\partial_x^{k+2} z^i\right\|^2 + \int_{\mathbb{R}} W(\partial_x^k E)^2 dx \\
=&(-1)^{\delta_{0k}} \sum_{i=a,b} \int_{\mathbb{R}} \partial_x^{(k-1)(1-\delta_{0k})}(f_{i,1} + f_{i,2})\partial_x^{k+2-\delta_{0k}} z^i dx \\
&- (1-\delta_{0k}) \sum_{i=a,b}\sum_{j=0}^{k-1} C_k^j \int_{\mathbb{R}} \partial_x^{k-j-1}(p''(W)W_x)\partial_x^{j+1} z^i \partial_x^{k+1} z^i dx \\
&+ \frac{1}{2}(-1)^k(\delta_{0k} + \delta_{1k}) \int_{\mathbb{R}} (z^a - z^b) E \partial_x^{k+k+1} z^a dx \\
&- (\delta_{2k} + \delta_{3k}) \int_{\mathbb{R}} \partial_x^{k-2}((z_x^a - z_x^b)E)\partial_x^{k+2} z^a dx \\
&- (-1)^{\delta_{1k}} \int_{\mathbb{R}} \partial_x^{\delta_{3k}}(z_x^b E)\partial_x^{2k-\delta_{3k}} E dx - \sum_{j=0}^{k-1} C_k^j \int_{\mathbb{R}} \partial_x^{k-j} W \partial_x^j E \partial_x^k E dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant(\alpha_1+O(\delta^2))\sum_{i=a,b}\left\|\partial_x^{k+1+\min(k,1)}z^i\right\|^2+O(\delta_T+\delta)^2\sum_{i=a,b}\left\|\partial_x^{2+\max(k-1,0)}z^i\right\|^2\\
&\quad+O(\delta_T^2)\left\|\partial_x^{1+\max(k-1,0)}z^i\right\|^2+\alpha_2\sum_{i=a,b}\left\|\partial_x^{k+1}z^i\right\|^2\\
&\quad+O(\delta^2)\frac{k}{\max(k,1)}\sum_{i=a,b}\sum_{j=0}^{k-1}C_k^j\left\|\partial_x^{j+1}z^i\right\|^2+(\delta_{0k}+\delta_{1k})O(\delta_T^2)(\|E\|^2+\left\|\partial_x^{2k+1}z^a\right\|)\\
&\quad+(\delta_{2k}+\delta_{3k})\Big[O(\delta_T^2)(\|E\|^2+\delta_{3k}\|E_x\|^2)+\alpha_3\left\|\partial_x^{k+2}z^a\right\|^2\Big]\\
&\quad+\alpha_4\left\|\partial_x^kE\right\|^2+O(\delta^2)\sum_{j=0}^{k-1}C_k^j\left\|\partial_x^jE\right\|^2,
\end{aligned}
$$

where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$, α_j's are small positive constants such that for sufficiently small δ_T, δ and T , $p'(w)-\alpha_1-\alpha_2-O(\delta_T+\delta)^2>0$, $\frac{\varepsilon^2}{2}-\alpha_3-O(\delta_T^2)>0$ and $W-\alpha_4-O(\delta_T^2)>0$. Thus, it is easy to obtain the desired result (2.32) by considering, in turn, every $k=0,1,2,3$, respectively.

Now, we prove the second result (2.34). Taking the kth order derivative of the second equation in (2.29) with respect to the spatial variable x, and taking the L^2 inner product of the resulting equation with ∂_x^kE, we have for $k=0,1,2$, in view of (2.31) and Lemmas 2.4 and 2.10, that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\partial_x^kE(t)\right\|^2+\delta_{0k}\int_{\mathbb{R}}p'(W)E_x^2dx+\frac{\varepsilon^2}{2}\left\|\partial_x^{k+2}E\right\|^2+2\int_{\mathbb{R}}W(\partial_x^kE)^2dx\\
=&\delta_{1k}\int_{\mathbb{R}}p'(W)E_xE_{xxx}dx+\delta_{2k}\int_{\mathbb{R}}p''(W)W_xE_xE_{xxxx}dx-2(1-\delta_{0k})(-1)^k\int_{\mathbb{R}}WE\partial_x^{2k}Edx\\
&+2\delta_{1k}\int_{\mathbb{R}}W(\partial_x^kE)^2dx-2\delta_{2k}\int_{\mathbb{R}}WE_x\partial_x^3Edx+2\delta_{0k}\int_{\mathbb{R}}(z^a+z^b)EE_xdx\\
&-\delta_{1k}\int_{\mathbb{R}}(z^a+z^b)(E_xE_{xx}+EE_{xxx})dx+\delta_{2k}\int_{\mathbb{R}}(z^a+z^b)(3E_{xx}E_{xxx}+E_xE_{xxxx})dx\\
&-(-1)^k(1-\delta_{2k})\int_{\mathbb{R}}(f_{a,1}-f_{b,1}+f_{a,2}-f_{b,2})\partial_x^{2k+1}Edx\\
&+\delta_{2k}\int_{\mathbb{R}}(f_{a,1}-f_{b,1}+f_{a,2}-f_{b,2})_x\partial_x^4Edx\\
\leqslant&\delta_{1k}[O(1)\|E_x\|^2+\alpha_5\|E_{xxx}\|^2]+\delta_{2k}[O(\delta^2)\|E_x\|^2+\alpha_6\|E_{xxxx}\|^2]\\
&+(1-\delta_{0k})[O(\delta^2)\|E\|^2+\alpha_7\left\|\partial_x^{2k}E\right\|^2]+\delta_{2k}O(\delta^2)[\|E_x\|^2+\|E_{xxx}\|^2]\\
&+\delta_{0k}[O(\delta_T^2)\|E\|^2+\alpha_8\|E_x\|^2]+\delta_{1k}O(\delta_T^2)[\|E_x\|^2+\|E_{xx}\|^2]+\delta_{1k}[O(\delta_T^2)\|E\|^2+\alpha_9\|E_{xxx}\|^2]\\
&+\delta_{2k}O(\delta_T^2)[\|E_{xx}\|^2+\|E_{xxx}\|^2]+\delta_{2k}[O(\delta_T^2)\|E_x\|^2+\alpha_{10}\|E_{xxxx}\|^2]\\
&+\delta_{0k}[O(\delta+\delta_T)\|E_x\|^2+O(\delta+\delta_T)^2\|E_x\|^2+\alpha_{11}\|E_{xx}\|^2]\\
&+\delta_{1k}[O(\delta+\delta_T)^2(\|E_x\|^2+\left\|E_{xx}^2\right\|)+\alpha_{12}\|E_{xxx}\|^2]
\end{aligned}
$$

$$+\delta_{2k}[O(\delta+\delta_T)^2(\|E_{xx}\|^2+\|E_{xxx}^2\|)+\alpha_{12}\|E_{xxxx}\|^2],$$

for some sufficiently small constants α_j's. Thus, by considering every $k=0,1,2$ in turn, there exist $\beta_0>\beta_1,\beta_2>0$ for δ and T small enough such that

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_x^k E(t)\right\|^2+\beta_k\left\|\partial_x^k E\right\|^2\leqslant(1-\delta_{0k})O(\delta_T+\delta)\|E\|^2,\quad k=0,1,2.$$

From Gronwall inequality, we can obtain

$$\left\|\partial_x^k E(t)\right\|\leqslant O(\delta_0+\delta)e^{-\beta_k t},\quad k=0,1,2.$$

Finally, let us prove the last result (2.33). Taking the kth order derivative of the first equation in (2.29) with respect to the spatial variable x, and taking the L^2 inner product of the resulting equation with $(1+t)^k\partial_x^k z^i$, we get for $k=1,2,3$, with the help of (2.31), (2.32), (2.34) and Lemma 2.4 and 2.10, that

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{(1+t)^k}{2}\sum_{i=a,b}\left\|\partial_x^k z^i\right\|^2\right]+(1+t)^k\sum_{i=a,b}\int_{\mathbb{R}}p'(W)(\partial_x^{k+1}z^i)^2dx\\
&\qquad+\frac{\varepsilon^2}{2}(1+t)^k\sum_{i=a,b}\left\|\partial_x^{k+2}z^i\right\|^2\\
=&\frac{k}{2}(1+t)^{k-1}\sum_{i=a,b}\left\|\partial_x^k z^i\right\|^2-(1+t)^k\sum_{i=a,b}\sum_{j=0}^{k-1}C_k^j\int_{\mathbb{R}}\partial_x^{k-j}p'(W)\partial_x^{j+1}z^i\partial_x^{k+1}z^i dx\\
&-2(1+t)^k\int_{\mathbb{R}}W(\partial_x^k E)^2dx-(1+t)^k\sum_{i=a,b}\sum_{j=0}^{k-1}C_k^j\int_{\mathbb{R}}\partial_x^{k-j}(z_x^i+W)\partial_x^j E\partial_x^k E dx\\
&-(1+t)^k\sum_{i=a,b}\int_{\mathbb{R}}z_x^i(\partial_x^k E)^2dx+(1+t)^k\sum_{i=a,b}\int_{\mathbb{R}}\partial_x^{k-1}(f_1^i+f_2^i)\partial_x^{k+2}z^i dx\\
\leqslant&\frac{k}{2}(1+t)^{k-1}\sum_{i=a,b}\left\|\partial_x^k z^i\right\|^2+\delta_{1k}(1+t)^k\sum_{i=a,b}\left[\alpha_{14}\left\|z_{xxx}^i\right\|^2+O(1)(p'(W))^2\left\|z_x^i\right\|^2\right]\\
&+\delta_{2k}(1+t)^k\sum_{i=a,b}\left[O(\delta^2)\left\|z_{xx}^i\right\|^2+\alpha_{15}\left\|z_{xxxx}^i\right\|^2+O(1)\left\|z_{xxx}^i\right\|^2\right]\\
&+\delta_{3k}(1+t)^k\sum_{i=a,b}\left[O(\delta^2)\left(\left\|z_{xx}^i\right\|^2+\left\|z_{xxx}^i\right\|^2\right)+O(1)\left\|z_{xxxx}^i\right\|^2+\alpha_{16}\left\|z_{xxxxx}^i\right\|^2\right]\\
&+O(\delta+\delta_T)O(\delta+\delta_0)^2,
\end{aligned}$$

which implies the desired result as the way as the previous. □

The proof of Theorem 1.2 and Proposition 2.9. Since the proof is very similar to the unipolar case in view of Lemma 2.11, we omit details. □

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